Adaptive variable structure hierarchical fuzzy control for a class of high-order nonlinear dynamic systems

Mohammad Mansouri a,*, Mohammad Teshnehlab b, Mahdi Aliyari Shoorehdeli c

a Department of Control Engineering, Faculty of Electrical Engineering, K.N. Toosi University of Technology, P.O. Box 16315-1355, Tehran, Iran
b Industrial Control Center of Excellence, Faculty of Electrical Engineering, K.N. Toosi University of Technology, Tehran, Iran
c Department of Mechatronics Engineering, Faculty of Electrical Engineering, K.N. Toosi University of Technology, Tehran, Iran

1. Introduction

The application of fuzzy systems to automation processes provides insight into the structure of control systems for control of high-order nonlinear dynamic systems. Hence, it is challenging issue due to its applications and there are significant literatures published in this area. Generally, a typical fuzzy system with n input variables and m membership functions for each variable, requires m^n rules. This phenomenon, increasing the number of rules with increasing of number of inputs exponentially, is called "curse-of-dimensionality". So, the curse of dimensionality comes to be as a serious problem to fuzzy systems' extensive application when they are applied to high-dimensional systems. To overcome this problem, Raju and Zhou [1] suggested hierarchical fuzzy systems. In hierarchical fuzzy systems (HFSs), instead of using the standard high-dimensional fuzzy system several lower dimensional fuzzy subsystems are arranged in the hierarchical structure. It makes the number of fuzzy rules grow linearly according to the number of inputs [2]. It has been proven that HFSs are universal estimators [2–5]. Refs. [6,7] are review papers introduced main topics of hierarchical fuzzy systems.


Based on the universal approximation property and in order to reach on-line learning ability in fuzzy systems, a number of stable...
adaptive fuzzy control schemes have been developed. The stability study in such schemes is carried out by using the Lyapunov design approach. There are two distinct methods in designing of a fuzzy adaptive control system; direct and indirect approaches. In the direct scheme, the fuzzy system is used for approximating an unknown ideal controller [17–19]. On the other hand, the indirect scheme uses fuzzy systems to estimate the plant dynamics and then produces a control law based on these estimations [20–22].

Till now, there is no investigation on the subject of Lyapunov based adaptive hierarchical fuzzy control due to difficulty in deriving the adaptation laws for the parameters appearing in the nonlinearly parameterized form. One notable paper is [8] where hierarchical fuzzy CMAC is used as Lyapunov based indirect adaptive controller. In the mentioned paper the HFCMAC is used for estimating the unknown nonlinear function \( f(X) \) and control gain \( b(X) \). However in our work we approximate two unknown nonlinear functions; \( f(X)/b(X) \) and \( (1/b(X)) \). The main advantage of our work with respect to [8] is that we consider the error of all tunable parameters of HFS as terms of the proposed Lyapunov function which guarantees convergence of them to their optimal values while in [8] just the error of the last layer's parameter of hierarchical fuzzy CMAC is considered in the Lyapunov function. Notice that in [29] the sliding surface is produced in hierarchical manner and adaptive fuzzy system is used for approximating unknown nonlinear function, but in this study adaptive hierarchical fuzzy system is investigated and the sliding surface is used for convergence of errors and it is not in the hierarchical structure. So, this work is completely different from [29].

In this study, a new adaptive variable structure hierarchical fuzzy control (VSHFC) system is presented. The proposed hierarchical fuzzy control system is able to reduce the number of rules with respect to the ordinary fuzzy control system. Some tunable parameters in the hierarchical fuzzy control (HFC) systems appear nonlinearly. So, the determination of the adaptive laws for such parameters is a nontrivial task. In this study, first a new description of the HFS for being a function approximator is presented. Then, this study offers a novel adaptive control method to tune all parameters of conclusion parts of fuzzy blocks located in different layers of the HFC. Taylor series expansion is applied for parameters which appear nonlinearly. The fuzzy switching system decides the operation area of the HFC and the variable structure control. Global boundedness of the overall adaptive system and the desired precision are established with the new adaptive control system using the Lyapunov-based stability theorem. In order to prove the proposed theorems, nonlinear systems are categorized into two categories; in the first category, the control gain is considered as unity and in the other category it is considered as non-unity. The theorems are proved for both categories. The proposed adaptive control system decreases the number of rules and consequently the number of tunable parameters with respect to the ordinary fuzzy control system. Finally, results of proposed theorems are simulated on four systems. The systems are categorized as robotics, mechanical, mathematical and physical systems.

It is important to notice that the proposed VSHFC system described in this paper differs from those described in the literature from these aspects: (i) For the first time, the HFC system is combined with the variable structure control. Also, its nonlinearly appearing adjustable parameters are tuned using adaptation laws achieved by the Lyapunov approach. Then the driven adaptation laws are made robust using \( \sigma \)-modification. (ii) There is an extra term in the Lyapunov approach to overcome the bounded disturbances of nonlinear systems. (iii) In order to gain soft switching system, it is considered as a fuzzy system.

The rest of this paper is organized as follows. Problem formulation is given in Section 2. Section 3 presents the formulation of the hierarchical fuzzy system as function approximators. Also, a theorem for describing the function approximation errors is developed and proven. In Section 4, the scheme of VSHFC system is proposed to control of SISO nonlinear systems. In addition, according to what is achieved from the theorem (presented in Section 3), two theorems and their conditions are presented and proven to ensure the asymptotically stability of nonlinear systems. Simulation results are provided in Section 5 to demonstrate the performance of the proposed method. The paper is discussed and concluded in Section 6.

2. Problem formulation

This study focuses on the design of adaptive control algorithms for a class of dynamic SISO affine nonlinear systems which can be expressed in the canonical form as follows:

\[
x^{(n)}(t)+f(x(t), x(t), \ldots, x^{(n-3)}(t)) = b(x(t), x(t), \ldots, x^{(n-1)})u(t)+d(t)
\]

where \( u(t) \) is a control input, \( f(X) \) is an unknown function, \( b(X) \) is the control gain and \( d(t) \) is the bounded disturbance such that \( |d(t)| < D \). The control objective is to force the states of the system, \( X = [x_1, x_2, \ldots, x_{n-1}] \), to follow a specified desired trajectory, \( X_d = [x_{d1}, x_{d2}, \ldots, x_{dn-1}] \). Considering \( e = X - X_d \) as the tracking error vector, our goal is to design a suitable control law, \( u(t) \), which ensures that \( e \to 0 \) as \( t \to \infty \).

In this paper, the hierarchical fuzzy system (HFS) is utilized to provide the control effort. The majority of solutions using adaptive control approaches has focused on the situation where an explicit linear parameterization of the unknown function is possible. Although, it is unfeasible for HFSs. Thus, the challenge addressed in this paper is development of adaptive HFCs where the explicit linear parameterization of the adaptive control system is impossible.

In the next Section, the hierarchical fuzzy system is introduced to approximate functions. In addition, the novel description of the function approximation errors in HFS is proposed.

3. Hierarchical fuzzy system as a function approximator

In this section, the HFS is presented as a function approximator. First, the structure of HFS is given. Then, Theorem 1 is proposed to describe the function approximator errors in HFS.

3.1. Hierarchical fuzzy system

In this section, the hierarchical fuzzy system considered in this study is introduced. Fig. 1 shows a typical HFS with \( n-1 \) layers where FIS is used for representation of fuzzy inference system. As shown in Fig. 1, every layer consists of one block which is an ordinary fuzzy system with two inputs. It is proven that the
hierarchical fuzzy system with this structure can be minimized the number of rules [3].

Generally, the output of the p-th block with employing the zero-order Takagi-Sugeno fuzzy system with Gaussian membership functions for inputs, singleton fuzzifier, sum-product inference engine, and center average defuzzifier is as follows:

\[ y_p = \sum_{k_p=1}^{n_{p+1}} \mu_{c_k}(x_{p+1}) \mu_{c_k}(y_{p-1}) \theta_{k_p}^T \]

(2)

where \( x_{p+1} \) is the \((p+1)\)-th input, \( y_{p-1} \) is the output of the \((p-1)\)-th layer, \( \theta_{k_p} \) is the center of the consequent part membership functions, \( \mu_{c_k}(\cdot) \) are the membership functions of \( x_{p+1} \) and \( y_{p-1} \) respectively and \( n_p \) is the number of rules for \( p \)-th block. It is necessary to indicate that for the first layer, \( y_0 \) is considered as \( x_2 \). It is possible to rewrite (2) in vector form as shown in (3).

\[ y_p = \theta_1^T \Gamma_p(x_{p+1}, y_{p-1}), \quad p = 1, \ldots, n-1 \]

(3)

where \( \theta_1 \in \mathbb{R}^{n \times n_p} \) is a vector of centers of consequent parts and \( \Gamma_p \in \mathbb{R}^{n \times n_p} \) is a vector of the normalized part due to the antecedent part of \( \Gamma_p \).

Please notice that centers of consequent parts of previous layers appear nonlinearly in the \( \Gamma_p \).

3.2. Novel description of the function approximation error in HFS

In this section, it is assumed that the HFS is used for function approximation and a novel description of estimation error is presented. Assume a function approximation error is defined as follows:

\[ e(X) = f(X) - \tilde{f}(X) \]

(4)

where \( \tilde{f}(X) \) is the estimated output of an HFS approximation of \( f(X) \) as described in (3) and \( \Gamma_1 \) with for \( i = 1, \ldots, n-1 \) is the vector of centers of consequent parts of \( i \)-th block. The parameters in the estimation should then be tuned to provide the effective function approximation architecture. The error between the ideal HFS and the function \( f(X) \) is defined as:

\[ e_1(X) = f(X) - f^*(X) = f(X) - \theta_1^T \Gamma_{n-1} \theta_1 \]

(5)

where \( \theta_1 \in \mathbb{R}^{n \times n_p} \) is an unknown constant vector composed of optimal weight matrices and some bounded constants and \( Y_i = \{1, |\theta_1|, |\theta_2|, \ldots, |\theta_i|\} \) is a known function vector.

**Proof 1.** Using (5), the function approximation error in (4) can be written as:

\[ e(X) = e_1(X) + \Gamma_{n-1} \Gamma_{n-1}^T e(X) + \theta_1^T \Gamma_{n-1} \theta_1 \]

(9)

So, the function approximation error can be written as follows:

\[ e(X) = e_1(X) + \theta_1^T \Gamma_{n-1} \theta_1 \]

In order to obtain a mathematical expression for \( \Gamma_{n-1} \), the Taylor’s series expansion of \( \Gamma_{n-1} \) is performed about \( \theta^* \) for \( i = 1, \ldots, n-2 \). This leads to

\[ \Gamma_{n-1}(X, \theta_1, \ldots, \theta_{n-2}) = \Gamma_{n-1} + \sum_{i=1}^{n-2} \frac{\partial^2 \Gamma_{n-1}}{\partial \theta_i^2} \theta_i + H.O.T \]

(11)

**Theorem 1.** Let us define the estimation errors of parameter vectors as:

\[ \theta_i = \theta_i^* - \theta_i, \quad i = 1, \ldots, n-1 \]

(6)

the function approximation error \( e(X) = f(X) - \tilde{f}(X) \), when the HFS with \((n-1)\) layers, \( n \) is the dimension of the state vector \( X \),are used for function estimation and all membership functions are considered as Gaussian functions, can be expressed as:

\[ e(X) = \sum_{i=1}^{n-2} \frac{\partial^2 \Gamma_{n-1}}{\partial \theta_i^2} \theta_i + \sum_{i=1}^{n-2} \frac{\partial^2 \Gamma_{n-1}}{\partial \theta_i^2} \theta_i + R \]

(7)

where \( R \) is a residual term which satisfies:

\[ |R| \leq \theta_i^T Y_i \]

(8)

where \( \theta_i^* \in \mathbb{R}^n \) is an unknown constant vector composed of optimal weight matrices and some bounded constants and \( Y_i = \{1, |\theta_1|, |\theta_2|, \ldots, |\theta_i|\} \) is a known function vector.
boundness. It is obvious that

\[ \| \theta_i^T \| \leq \delta_i, \quad i = 1, \ldots, n - 1 \] (18)

The term \( R \) in (15) can be bounded as follow:

\[ |R| = \left| \epsilon_f(x) + \partial^T \sum_{i=1}^{n} \frac{\partial \Gamma_{n-i}}{\partial \theta_i} + \epsilon_{f}^T (H.O.T.) \right| \] (19)

Considering the fact that \( |\epsilon_f(x)| \leq \epsilon^* \), where \( \epsilon^* \) is a constant and using (17), (18) and (19) can be written as:

\[
\begin{align*}
|\epsilon| & \leq \epsilon^* + \sum_{i=1}^{n} C_i \partial_i \\
& + \sum_{i=1}^{n} C_i (\| \partial_i + \partial_i \|) \\
& = \epsilon^* + 2 \sum_{i=1}^{n} C_i \partial_i \\
& + \sum_{i=1}^{n-2} C_i \partial_i + \sum_{i=1}^{n} C_i (\| \partial_i \|)
\end{align*}
\]

Finally, it is concluded that

\[ |R| \leq \left[ \theta_{n-1}, \theta_{n-2}, \ldots, \theta_1 \right]^T 1, \| \partial_{n-1} \|, \| \partial_{n-2} \|, \ldots, \| \partial_1 \|]^T = \sigma_{f}^T Y_f 
\]

where

\[
\begin{align*}
\theta_{n-1} & = 2 \sum_{i=1}^{n-2} C_i \partial_i + \epsilon^* \\
\theta_{n-2} & = \sum_{i=1}^{n-2} C_i \partial_i \\
\theta_1 & = C \partial_1, \quad i = 1, \ldots, n - 2
\end{align*}
\]

**Remark** Theorem 1 expresses the high order terms of Taylor series expansion as a linearly parameterizable structure with respect to the residual term. In addition, the residual term is bounded by a linear expression with a known vector of functions. On the other hand, there is no explicit expressions for optimal parameters. Therefore, the linearly parameterized form of the residual term makes it possible to develop the stable learning method using adaptive control techniques.

In the next section, what is achieved from Theorem 1 is applied to develop the VSHF control system for a class of high-order nonlinear dynamic systems.

### 4. Adaptive variable structure hierarchical fuzzy control

In this Section, VSHFC system is proposed to control of SISO nonlinear systems. First, the structure of VSHFC is presented. Second, it is utilized to control of the nonlinear systems. Two theorems are proven to ensure the asymptotically stability of nonlinear systems described in (1) with considering \( b(X) = 1 \) or \( b(X) \neq 1 \).

#### 4.1. Structure of VSHF control system

The schematic diagram of the proposed VSHFC system is shown in Fig. 2. This structure differs from those described in [23] where the RBF neural network is used instead of the HFC presented here. As it is shown in the figure, the control signal composed of three terms: (1) linear feedback, (2) sliding mode, (3) HFS. The modulate block plays the role of weighting function which produces the appropriate value between zero and one based on the error for the variable structure control and hierarchy fuzzy control systems.

The switching surface considered as an error metric is defined as:

\[
s(t) = \left( \frac{d}{dx} + \lambda \right)^{n-1} e(t) \quad \text{with} \quad \lambda > 0
\]

which can be rewritten as \( s(t) = \Lambda^T e(t) \) with \( \Lambda^T = [\lambda^1, \ldots, \lambda^{n-1}, (n-1)\lambda^{n-2}, \ldots, 1] \). The equation \( s(t) = 0 \) demonstrates a time-varying hyperplane in \( \mathbb{R}^n \) such that the tracking error vector decreases exponentially to zero on it, so that perfect tracking can be asymptotically achieved by keeping up this condition [25]. Therefore, the control objective becomes the design of a controller to forces \( s(t) = 0 \). The time derivative of switching surface can be written as:

\[
\dot{s}(t) = -\lambda_2 s_2(t) + \lambda_1 e(t) + bu - f(X) + d(t)
\]

where \( \lambda_1 = (0, \lambda^{n-1}, (n-1)\lambda^{n-2}, \ldots, (n-1)\lambda) \). The objective of this paper is determination of adaption laws for \( \partial_i \), \( i = 1, \ldots, n - 1 \) using the error Eq. (23) such that all signals remain bounded and \( s(t) = 0 \).

Main results are presented in two steps. In the first stage, the control gain considered as unity i.e. \( b(X) = 1 \) and the global boundness of all system signal is established. This lets us concentrate on the structure of the adaptive controller, which stably tunes all parameters in the HFC which appear in the nonlinearly parameterized form. In the second stage, results are extended to the case of the non-unity control gain i.e. \( b(X) \neq 1 \). This gives a complete solution for the control problem.

#### 4.2. Asymptotically stability of nonlinear systems with considering \( b(X) = 1 \)

The stability of the closed-loop system when \( b(X) = 1 \) is established in the following theorem.

**Theorem 2.** If adaptive control laws, presented in by defining the switching surface presented in (29), and considering the parameter adjust rules given in (30) to (33) are applied to the nonlinear dynamic system introduced in (1) with \( b(X) = 1 \) (according to Fig. 2), then all states in the adaptive system will remain bounded, and the tracking errors will be asymptotically bounded by \( |e(t)| \leq 2^{n} \lambda^{n+1} \phi \), for \( i = 1, \ldots, n - 1 \).

\[
u(t) = -k_s s(t) + u_i(t) + (1 - m(t))u_{f}(t) + m(t)u_{d}(t) + u_i(t)
\]

\[
u_{f}(t) = \lambda_1 s(t) - \Lambda_1 e(t)
\]

\[
u_{d}(t) = \Theta_{n-1} \dot{r}_{n-1} - \dot{r} \Theta_{f} Y_f \text{sat}(s(t)/\phi)
\]

\[
u_{i}(t) = -k_i \text{sat}(s(t)/\phi)
\]

\[
u_{e}(t) = -\rho \text{sign}(s(t))
\]

\[
u_{d}(t) = s(t) - \phi \text{sat}(s(t)/\phi)
\]
\[ \dot{\theta}_{n} = (1-m(t))\dot{s}_{g}(t) + \frac{n+2\sum_{i=1}^{n-1}\Gamma_{i}^{-1}(\theta - \dot{\theta}_{n})}{\partial_{\theta}^{T}\Gamma_{n-1}} \]  
\[ \dot{\theta}_{i} = (m(t)-1)\dot{s}_{g}(t)P_{n}^{-1}\left(\partial_{\theta}^{T}\Gamma_{n-1} \theta = \theta \right)^{T} ; j = 1, \ldots, n-2 \]  
\[ \dot{\theta}_{f} = (1-m(t))\dot{\theta}_{f}P_{n}^{-1}\Gamma_{n}^{T}Y_{r} \]  
where \( m(t) \) is the output of the zero-order Takagi-Sugeno fuzzy system with singleton fuzzifier, sum-product inference engine, and center average defuzzifier fuzzy system whose input is just \( s(t) \) and it has two Gaussian membership functions for the input variable and its value is \( 0 < m(t) < 1 \). Also, \( P_{n}^{-1} = \epsilon R^{n \times n} \), \( i = 1, \ldots, n-1 \) where \( r_{i} \) is the dimension of \( \theta_{i} \) and \( P_{n}^{-1} = \epsilon R^{n \times n} \). The \( sat(\cdot) \) function is defined as follows:

\[ sat(y) = \begin{cases} 
-1 & \text{if } y < -1 \\
1 & \text{if } y > 1 \\
y & \text{if } -1 \leq y \leq 1 
\end{cases} \]  

**Proof 2.** With substituting (24) in (23), \( \dot{s}(t) \) can be written as:

\[ \dot{s}(t) = -k_{d}\dot{s}(t) + (1-m(t))u_{df}(t) + m(t)u_{df}(t) - f(X) + u_{dc}(t) + d(t) \]

Using (26) and (7), \( u_{df}(t) \) can be written as:

\[ u_{df}(t) = f(X) - \epsilon X - \dot{\theta}_{f}^{T}Y_{r}sat\left(\frac{\theta_{r}}{\epsilon}\right) = f(X) + \dot{\theta}_{f}^{T}\left(\sum_{i=1}^{n-1}\Gamma_{i}^{-1}(\theta - \dot{\theta}_{n-1}) \right) - \frac{\theta_{r}}{\epsilon} \]

\[ \dot{\theta}_{n-1}^{T} \left(\sum_{i=1}^{n-2}\Gamma_{i}^{-1}(\theta - \dot{\theta}_{n-1}) \right) = \frac{\theta_{r}}{\epsilon} \]

Eq. (35) can be expressed as:

\[ \dot{s}(t) = -k_{d}\dot{s}(t) + (1-m(t))\dot{\theta}_{f}^{T}Y_{r}sat\left(\frac{\theta_{r}}{\epsilon}\right) \]

Using (26) and (7), \( u_{df}(t) \) can be written as:

\[ u_{df}(t) = f(X) - \epsilon X - \dot{\theta}_{f}^{T}Y_{r}sat\left(\frac{\theta_{r}}{\epsilon}\right) = f(X) + \dot{\theta}_{f}^{T}\left(\sum_{i=1}^{n-1}\Gamma_{i}^{-1}(\theta - \dot{\theta}_{n-1}) \right) - \frac{\theta_{r}}{\epsilon} \]

\[ \dot{\theta}_{n-1}^{T} \left(\sum_{i=1}^{n-2}\Gamma_{i}^{-1}(\theta - \dot{\theta}_{n-1}) \right) = \frac{\theta_{r}}{\epsilon} \]

Consider the Lyapunov function candidate as:

\[ V(t) = \frac{1}{2}\dot{s}_{g}^{2}(t) + \dot{\theta}_{n-1}^{T}P_{n-1}\dot{\theta}_{n-1} + \dot{\theta}_{f}^{T}P_{n}^{-1}\dot{\theta}_{f} + \frac{1}{\epsilon^{2}}(\rho - \rho^{*})^2 \]

It is necessary to indicate that the last term in the Lyapunov function is just for designing \( u_{t} \) which is added to control signal in order to compensate the disturbance. Taking the derivative of (38) gives:

\[ \dot{V}(t) = s_{g}(t)\dot{s}_{g}(t) - \dot{\theta}_{n-1}^{T}P_{n-1}\dot{\theta}_{n-1} - \dot{\theta}_{f}^{T}P_{n}^{-1}\dot{\theta}_{f} + \frac{1}{\epsilon^{2}}(\rho - \rho^{*})^2 \]

Applying (24) to (32) (in 39) leads to:

\[ \dot{V}(t) = -k_{d}\dot{s}_{g}^{2}(t) + (1-m(t))\dot{s}_{g}(t)\dot{\theta}_{n-1}^{T} \left(\sum_{i=1}^{n-2}\Gamma_{i}^{-1}(\theta - \dot{\theta}_{n-1}) \right) - \frac{1}{\epsilon^{2}}(\rho - \rho^{*})^2 \]

With some mathematical simplification and using upper bounds in applying the triangular inequality and also using the fact that \( |\dot{\theta}_{f}^{T}Y_{r}| \), \( \dot{s}_{g}(t) \), \( sat(\dot{s}_{g}(t)/\theta_{r}) = |\dot{s}_{g}(t)| \) result in the following equation. It is necessary to mention that if \( |\dot{s}| \leq \theta \) then \( s_{0} = 0 \), while if \( |\dot{s}| > \theta \) then \( s_{0} = \theta \).

\[ V(t) = -k_{d}\dot{s}_{g}^{2}(t) + (1-m(t))\dot{s}_{g}(t)(u_{df}(t) - f(X)) - (1-m(t))\dot{s}_{g}(t)\dot{\theta}_{f}^{T} \]

\[ \frac{1}{\epsilon^{2}}(\rho - \rho^{*})^2 \]

Since \( \rho^{*} \) is a designing parameter, considering it as \( D \leq \rho^{*} \) gives:

\[ \dot{V}(t) = -k_{d}\dot{s}_{g}^{2}(t) + (1-m(t))\dot{s}_{g}(t)(u_{df}(t) - f(X)) - (1-m(t))\dot{s}_{g}(t)\dot{\theta}_{f}^{T} \]

\[ \frac{1}{\epsilon^{2}}(\rho - \rho^{*})^2 \]

Therefore, all signals in the system presented in (1) are bounded. Since \( s_{0}(t) \) is bounded, it can be shown that, if \( u(t) \) is bounded, then \( e(t) \) is also bounded for all \( t \) on the other hand, since \( x_{0}(t) \) is bounded by designing it, \( X(t) \) is as well. In order to investigate the asymptotic convergence of the tracking error, it is necessary to prove that \( s_{0}(t) \rightarrow 0 \) as \( t \rightarrow \infty \). This can be achieved by applying Barbalat’s Lemma to the continuous, non-negative function as follow:

\[ V_{1}(t) = V(t) - \int_{0}^{t}(V(t) + k_{d}\dot{s}_{g}^{2}(t))dt = V_{1}(t) = -k_{d}\dot{s}_{g}^{2}(t) \]

It can be easily observed that every term on the right-hand side of (35) is bounded, hence \( s_{0}(t) \) is bounded. This means that \( V_{1}(t) \) is a uniformly continuous function of time. Since \( V_{1}(t) \) is bounded below by 0, and \( V_{1}(t) \leq 0 \) for all \( t \), applying Barbalat’s lemma results in \( V_{1}(t) \rightarrow 0 \) as \( t \rightarrow \infty \). This implies that the inequality \( |s(t)| \leq \theta \) is asymptotically satisfied. Also, the asymptotic tracking errors as written in [25] are asymptotically bounded by:

\[ |e^{h}(t)| \leq 2^{i}x_{i}^{-1} \theta_{i} \]

**Remark 1.** Theorem 2 demonstrates that it is possible to set up a globally stable adaptive system by tuning the consequence parameters of all blocks of the HFS control system which are appeared nonlinearly in the system. This can be accomplished according to the proof of Theorem 2, due to the linear expression of \( \dot{\theta}_{i} \) for \( i = 1, \ldots, n-1 \) in Theorem 1.
Remark 2. Inequality for tracking errors confirms that the tracking error can be effected by $\phi$. If $\phi \to 0$, then $e(t) \to 0$. In such a case, $sat(s(t)/\phi)$ becomes $sign(s(t))$. In this way, the chattering phenomena can be overcome, which makes this method more applicable.

Remark 3. As mentioned in Theorem 2, $m(t)$ is the output of an ordinary fuzzy system. In fact, based on the value of switching surface which is the input of modulate fuzzy system, it produces weights for variable structure and HFS control systems. This gives the ability to the control system to switch softly between the two controllers.

Corollary 1. (Generalization). Adaptation laws are considered for HFS in the typical structure of Fig. 1. Notice that they can be applied to any HFS with any structure and any number of inputs at each block. In fact, the important thing is the number of layers appeared as the upper bound of summations at adaptation laws. If the number of layers are less than $-1$, then just the upper bound of summations should change to the number of layers. This can be more interesting when there is just one layer where the HFS convert to the ordinary fuzzy system. In the latter case, all summations at adaptation laws must be considered as zero. Therefore, the proposed method is more general than which is used in the ordinary fuzzy system and it can be applied to both of HFS (with any structure) and the ordinary fuzzy system.

4.3. Asymptotically stability of nonlinear systems with considering $b(X) \neq 1$

In Theorem 2, it is required that the control gain be unit. In the following theorem, the results are extended to nonlinear systems with non-unitary control gains.

Notice. The following general assumptions with respect to the control gain $b(X)$ are made:

- The control gain $b(X)$ is finite and non-zero.
- The functions $h(x) = f(x)/b(x)$ and $g(x) = 1/b(X)$ are bounded by known positive values such that $|h(x)| \leq M_0$ and $|g(x)| \leq M_1$.
- There exists a known positive function $M_2(x)$, such that $|d/dt(g(x)h(x))| \leq M_2(x)|x|$.

Let us denote $h(x) = h(x) = \hat{h} = \hat{h}_{h(h-1)} = \hat{h}_{h-1}, \hat{h}_{h-2}, \ldots, \hat{h}_{h-n-2}$ and $g(x) = \hat{g} = \hat{g}_{g(n-1)} = \hat{g}_{g(n-2)}, \ldots, \hat{g}_{g(0)}$ to be the estimates of the optimal fuzzy approximator $\hat{h}_{h(h-1)} = \hat{h}_{h-1}, \hat{h}_{h-2}, \ldots, \hat{h}_{h-n-2}$ and $\hat{g}_{g(n-1)} = \hat{g}_{g(n-2)}, \ldots, \hat{g}_{g(0)}$, respectively. Theorem 1 can be applied in order to obtain the following approximation error properties:

$$\dot{h} = h - \hat{h} = \hat{h}_{h(h-1)} = \hat{h}_{h-1} = \hat{h}_{h-2}, \ldots, \hat{h}_{h-n-2}$$

$$\dot{g} = g - \hat{g} = \hat{g}_{g(n-1)} = \hat{g}_{g(n-2)}, \ldots, \hat{g}_{g(0)}$$

Furthermore, $|\dot{h}| \leq \hat{h} | \hat{Y}_h$ and $|\dot{g}| \leq \hat{g} | \hat{Y}_g$. With these error features, the stability of the closed-loop system when $b(X) \neq 1$ is established in the following theorem.

Theorem 3. If adaptive control laws, presented in (46) to (48) by considering the parameter adjust rules given in (49) to (55), are applied to the nonlinear dynamic system introduced in (1) with $b(X) \neq 1$ and based on the assumptions mentioned above (according to Fig. 2), then all states in the adaptive system will remain bounded, and the tracking errors will be asymptotically bounded by $|e^{(i)}(t)| \leq 2 \frac{1}{\phi} \phi_i$, for $i = 1, \ldots, n-1$.

Proof. With substituting (46) into (23), $g(x)\dot{h}(t)$ can be expressed as:

$$g(x)\dot{h}(t) = -k_d s(t) - \frac{1}{2} M_2 XX S_x(t) + (1 - m(t)) u_a(t) + m(t) u_a(t) + u(t)$$

$$u(t) = -k_d s(t) - \frac{1}{2} M_2 XX S_x(t) + (1 - m(t)) u_a(t) + m(t) u_a(t) + u(t)$$

With substituting (44) and (45) into (47), it is concluded that

$$u(t) = -k_d s(t) sat \left( \frac{S(t)}{\phi} \right) + g(x)\dot{h}(t)$$

Please cite this article as: Mansouri M, et al. Adaptive variable structure hierarchical fuzzy control for a class of high-order nonlinear dynamic systems. ISA Transactions (2014), http://dx.doi.org/10.1016/j.isatra.2014.11.014
\[ (g(X) - \hat{g}(X)a_t - \hat{\theta}_Y Y_s[a_t] \sigma \frac{\text{sat}(s(t))}{\rho} = h(X) \]
\[ + {\hat{\theta}}_{h(n-1)} \left( \sum_{i=1}^{n-2} \frac{\partial^2 \hat{g}(n-1)}{\partial \theta_i^2} \vartheta \right) - \hat{\rho}_s \hat{y}_t \sigma \frac{\text{sat}(s(t))}{\rho} \]
\[ - \hat{\theta}_{Y_i} Y_{s}(a_t) \sigma \text{sat}(s(t)) \frac{1}{\rho} \]
\[ \nu \left( \sum_{i=1}^{n-2} \frac{\partial^2 \hat{g}(n-1)}{\partial \theta_i^2} \vartheta \right) + \hat{\rho}_s \hat{y}_t \sigma \text{sat}(s(t)) \frac{1}{\rho} \]
\[ \leq -k_{s \rho} \hat{y}_t (t) - \rho^* |s_{\phi}| + (\rho^* - \rho^*) |s_{\phi}| + |s_{\phi}| M_1 + \frac{1}{2} \rho (\rho - \rho^*) \]
\[ \leq -k_{s \rho} \hat{y}_t (t) - \rho^* |s_{\phi}| + D M_1 |s_{\phi}| \]
\[ \leq \rho^* |s_{\phi}| + (\rho^* - \rho^*) |s_{\phi}| + |s_{\phi}| M_1 + \frac{1}{2} \rho (\rho - \rho^*) \]
\[ \leq -k_{s \rho} \hat{y}_t (t) - \rho^* |s_{\phi}| + D M_1 |s_{\phi}| \]
\[ \leq \rho^* |s_{\phi}| + (\rho^* - \rho^*) |s_{\phi}| + |s_{\phi}| M_1 + \frac{1}{2} \rho (\rho - \rho^*) \]
\[ \leq -k_{s \rho} \hat{y}_t (t) - \rho^* |s_{\phi}| + D M_1 |s_{\phi}| \]

Corollary 2. (Generalization) The same as corollary 1 at Theorem 2, adaptation laws can be applied to any HFS with any structure and any number of inputs at each block especially to ordinary fuzzy systems. So, this theorem can be used in any case if h(x) or g(X) is HFS or the ordinary fuzzy system. Furthermore, the proposed theorem is general.

4.4 Robust adaptation laws

In the previous section, the only uncertainty of the dynamical system is considered in unknown controller parameters. However, in practical applications, when real systems describe with fuzzy models modeling errors are inevitable [26]. Also, there are other sources of modeling errors, such as unmodeled dynamics, measurement noise, time variation of parameters and disturbances [27]. Here, it is assumed that the HFS does not exactly describe the nonlinear functions leading to modeling error and a stability analysis is presented under this condition.

Considering Theorem 2, the adaptation laws of (30)-(33) are modified as follows

\[ \hat{\theta}_{n+1} = (1 - m(t)) \hat{\theta}_{n+1} + \left( \sum_{i=1}^{n-2} \frac{\partial^2 \hat{g}(n-1)}{\partial \theta_i^2} \vartheta \right) - \sigma P_{n-1} \hat{\theta}_{n-1} \]

\[ \hat{\rho}_s = \gamma |s_{\phi}| - \sigma \rho \]

where \( \sigma \) is a small positive number. Considering the same Lyapunov function as in (8) and following a similar analysis as in the proof of Theorem 2 one has

\[ V \leq -k_{s \rho} \hat{y}_t (t) - \rho^* |s_{\phi}| + \sum_{i=1}^{n-2} \frac{\partial^2 \hat{g}(n-1)}{\partial \theta_i^2} \vartheta \hat{\theta}_{n+1} - \sigma \rho (\rho - \rho^*) \]

Finally, it is concluded that

\[ V \leq -c V + \rho \]

where

\[ \rho = \frac{\sigma^2 \rho^* \rho^*}{2} + \frac{\sigma^2 \rho^* \rho^*}{2} + \sum_{i=1}^{n-2} \frac{\partial^2 \hat{g}(n-1)}{\partial \theta_i^2} \vartheta \hat{\theta}_{n+1} + \sigma \rho^* \rho^* \]

and

Please cite this article as: Mansouri M, et al. Adaptive variable structure hierarchical fuzzy control for a class of high-order nonlinear dynamic systems. ISA Transactions (2014), http://dx.doi.org/10.1016/j.isatra.2014.11.014
in which without loss of generality it is assumed that the matrices of $P_i^{-1}$ for $i = 1,...,n-1$ are diagonal matrices with the same diagonal elements as $\gamma_i$ for $i = 1,...,n-1$. Therefore, the Lyapunov function converges until $V \leq \rho_c / 2$.

Let us consider Theorem 3, the modified adaptation laws of (49)-(55) are as follows:

$$\dot{h}_{n(n-1)} = -(1-m(t))s_g(t)P_{n(n-1)}^{-1}\left(\frac{\partial^2 h_{n(n-1)}}{\partial \theta_i^T \partial \theta_i} \theta - \dot{\theta}_{n(n-1)} \right)$$

$$\dot{h}_{n(n-1)} = -(1-m(t))s_g(t)P_{n(n-1)}^{-1}\left(\frac{\partial^2 h_{n(n-1)}}{\partial \theta_i^T \partial \theta_i} \theta - \dot{\theta}_{n(n-1)} \right)$$

$$\dot{g}_{n(n-1)} = -(1-m(t))s_g(t)P_{n(n-1)}^{-1}\left(\frac{\partial^2 g_{n(n-1)}}{\partial \theta_i^T \partial \theta_i} \theta - \dot{\theta}_{n(n-1)} \right)$$

$$\dot{\gamma}_{n(n-1)} = -(1-m(t))s_g(t)P_{n(n-1)}^{-1}\left(\frac{\partial^2 \gamma_{n(n-1)}}{\partial \theta_i^T \partial \theta_i} \theta - \dot{\theta}_{n(n-1)} \right)$$

Finally, it is concluded that

$$V \leq -cV + \rho_c$$

where

$$\rho_c = \frac{\sigma}{2} \sum_{i=1}^{n-2} \left[ \frac{\theta_i^T \theta_i + \theta_i^T \theta_i - \theta_i^T \theta_i}{\theta_i^T \theta_i} \right]$$

and

$$c = \min(\sigma, 2k_\alpha) \min(\gamma, T_f, \gamma)$$

It is assumed that the matrices of $P_{n(n-1)}^{-1}$, $P_{n(n-1)}^{-1}$, $P_{n(n-1)}^{-1}$, $P_{n(n-1)}^{-1}$ for $i = 1,...,n-1$ are diagonal matrices with the same diagonal elements as $\gamma_i$ for $i = 1,...,n-1$. Hence, the Lyapunov function converges until $V \leq \rho_c / 2$.

The stable region in which the controlled system is stable after applying modifications is as follows:

$$\frac{1}{\rho_c} \left( V \right)^\frac{1}{2}$$

It is important to note that a high value for $\sigma$ would make the system more robust while a small value of $\sigma$ make the performance of the system better.

In the next section, simulation results are presented to more explain the proposed theorems.

5. Simulation results

In this section, four examples are illustrated; (1) a robotic system: the flexible joint robot, notice that in this case $b(X) = 1$. So, Theorem 2 is applied on it (2) a mechanical system: a quarter car active suspension system, in this case $b(X) = cte$ and Theorem 2 can be applied on it (3) a nonlinear system, notice that in the last cases $b(X) \neq 1$ So, Theorem 3 is applied on it (4) a physical system: the hyper chaos system, such that in this case $b(X) = 1$. So, Theorem 2 is applied on it. The last case is considered to reveal the efficiency of proposed theorems when HFSs are considered for the estimation of both of $h(X)$ and $g(X)$. The results are compared with the proposed method in [8].

It is necessary to notice that as mentioned in corollaries, theorems can be applied to the ordinary fuzzy system. Simulation results of the proposed method with ordinary fuzzy systems with the same initial condition and control parameters with HPS lead to the exactly same results. This case is shown in appendix A.3 for the first example.

Example 1. (for Theorem 2): Fig. 3 shows the arrangement of the flexible joint robot schematically. Define $q = \{q_1, q_2, q_3, q_4\}$ as the set of generalized coordinates for the system where $q_3$ is the angle of the link, $q_2 = -1/m \theta_1$ is the angular displacement of the motor, $m$ is the gear ratio and $q_3 - q_2$ is the elastic displacement of the link, using Euler–Lagrange equations, the analytical model of the flexible joint robot is derived as [26]:

$$\begin{cases}
\dot{q}_1 + Mg \sin(q_1) + K(q_1 - q_2) = 0 \\
\dot{q}_2 - K(q_1 - q_2) = u
\end{cases}$$

The state space representation of the system is derived as:

$$\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} = A(x)\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} + B(x)u_2$$
The mechanism of a quarter car suspension system.

The structure of hierarchical fuzzy controller used in the example.

Table 1
Numerical values of parameters considered in simulation.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Numerical value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( g )</td>
<td>( 9.8 \text{ m/s}^2 )</td>
</tr>
<tr>
<td>( M )</td>
<td>( 1 \text{ kg} )</td>
</tr>
<tr>
<td>( K )</td>
<td>( 1 \text{ N/m} )</td>
</tr>
<tr>
<td>( f )</td>
<td>( 1 \text{ K.g/m} )</td>
</tr>
<tr>
<td>( l )</td>
<td>( 1 \text{ m} )</td>
</tr>
</tbody>
</table>

From (86) and Table 1, it is obvious that in this example \( f(Z) \) and \( h(Z) \) are defined as

\[
f(Z) = (9.8 \cos(z_1) + 2)z_1 - 9.8(z_1^2 - 1) \sin(z_1)
\]

(88)

The structure of the hierarchical fuzzy controller applied in this example is depicted in Fig. 4. The numerical values of controller’s parameters achieved by trial and error are shown in Table 2.
parameters are chosen randomly. Also, the initial values of states are considered as follows; $X(0) = [1 \ 0 \ 0 \ 0]^T$ for the sinusoidal signal and $X(0) = [1 \ 0 \ 0 \ 0]^T$ for the pulse signal. Also, in this example the description of fuzzy switching system, $m(t)$, and the rule base of HFS are expressed in the appendix for more clarification.

In tracking of the sinusoidal signal, after few seconds the response of the controlled signal follows the sinusoidal signal because the desired signal has the smooth behavior. Although, in tracking of the pulse signal the sudden change in the desired signal causes it takes time for the adaptation of tunable parameters. As it is depicted in Fig. 4, the control performance of the proposed adaptive control scheme has been demonstrated in simulations and the proposed control system makes the system states track its desired trajectories. Also, the results reveals the performance of proposed method in comparison with the method in [8].

**Example 2. (for Theorem 2):** In this example, the regulation problem of a quarter car active suspension system given by (89) is considered [28]. Fig. 6 schematically shows the mechanism of mentioned system.

$$
\dot{x}_1 = x_2 \\
\dot{x}_2 = \frac{1}{m_u}( - k_s(x_1 - x_3) - B_s(x_2 - x_4)^3 + u(t)) \\
\dot{x}_3 = x_4 \\
\dot{x}_4 = \frac{1}{m_u}( k_s(x_1 - x_3)^3 + B_s(x_2 - x_4)^5 - k_t x_3 + K_t Z_r - u(t)) + d(t)
$$

where $X_1$, $x_2$ are displacement and vertical velocity of sprung mass and $x_3$, $x_4$ are displacement and vertical velocity of unsprung mass, respectively. $k_s$ is the stiffness of the suspension, $B_s$ is the damping of the suspension, $k_t$ is the stiffness of the tire, $Z_r$ is the road displacement input, $m_u$, $m_s$ are sprung and unsprung masses respectively and $U$ is the active control. Also, $n(t)$ is a white noise signal with mean parameter of zero and standard deviation of 0.1 and $d(t)$ is defined in (90). The numerical values of the system’s parameters considered in the simulation depicted in Table 3.

$$
d(t) = \begin{cases} 
0.08 \cos(8 \pi t), & 1 < t < 1.2 \\
0, & \text{otherwise}
\end{cases}
$$

The state space form of (89) is not in the normal form. So, it does not satisfy the conditions stated in (1). With changing the coordinates of the system such that they satisfy the conditions of (1), the resulting system will be in the following form

$$
\dot{z}_1 = z_2 \\
\dot{z}_2 = z_3 \\
\dot{z}_3 = z_4 \\
\dot{z}_4 + f(Z) = b u(t) + d(t)
$$

where

$$
f(Z) = \frac{k_t}{m_u}k_s(x_1 - x_3)^3 + B_s(x_2 - x_4)^5 - k_t
$$

$$
b = \frac{k_t}{m_u}
$$

with

$$
X_1 = \frac{1}{m_u}( z_1 + \frac{m_s z_3}{k_t}) \\
X_2 = \frac{1}{m_u}( z_2 + \frac{m_s z_4}{k_t}) \\
X_3 = \frac{z_3}{k_t} \\
X_4 = \frac{z_4}{k_t}
$$

Eq. (91) indicates that the resulting system is in the form of (1). The structure of the HFC control system applied in the example is depicted in Fig. 7. It is important to note that in this example control gain is not unity but it is constant and Theorem 2 can be applied to it. Numerical values of parameters of the control system are shown in Table 4. These values are achieved by trial and error.

The output of system must approach to zero in this example to show efficiency of the proposed controller in regulation problems. Also, three membership functions are considered for each input of each block in every simulation. Thus, every block has 9 rules and the total number of rules in HFS will be 27. On the other hand, the total number of rules and the number of tunable parameters in comparison with the ordinary fuzzy control system decrease from 81 to 27. Initial values of tunable parameters are randomly chosen. Also, the initial values of states are considered as follows; $X(0) = [0.1 \ 0 \ 0 \ 0]^T$. Simulation results for regulation of a quarter car suspension system are shown in Fig. 8.

The regulation problem of the output of quarter car suspension which is displacement of sprung mass is done well. As it is
depicted in Fig. 8, the suggested controller force the system to track the desire constant signal with no overshoot or undershoot. Therefore, using proposed control technique regulation problem of the typical real world examples is properly possible. Also, comparison of the results and the method in [8] shows the performance of proposed method.

**Example 3.** (for Theorem 3): In this example, a nonlinear system given by (95) is considered.

\[
\begin{align*}
X_1 &= x_2 \\
X_2 &= x_3 \\
X_3 &= x_4 \\
X_4 + f(X) &= b(X)u(t) + d(t) \\
y &= x_1 + n(t)
\end{align*}
\]  

(95)

where

\[
\begin{align*}
f(X) &= \frac{-1}{1 + x_1^2 + x_2^2 + x_3^2 + x_4^2} \\
b(X) &= \frac{1}{1 + \sin(x_1)\cos(x_2) + \sin(x_3)} \\
d(t) &= 0.1 \sin(t)
\end{align*}
\]  

(96) (97) (98)

Where \(n(t)\) is a white noise signal with mean parameter of zero and standard deviation of 0.1. It is obvious that the nonlinear system of (95) is in the form described in (1). Also, it can be easily shown that the system in (95) satisfies all assumptions which are essential for applying Theorem 3. Structures of HFC systems applied in the example for the estimation of functions \(h(X) = f(X)/b(X)\) and \(g(X) = 1/b(X)\) are shown in Fig. 9. Numerical values of parameters are shown in Table 5.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Numerical values</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\phi)</td>
<td>0.0005</td>
</tr>
<tr>
<td>(k_d)</td>
<td>10</td>
</tr>
<tr>
<td>(k_{ad})</td>
<td>1</td>
</tr>
<tr>
<td>(\lambda)</td>
<td>3</td>
</tr>
</tbody>
</table>

Like previous example, three membership functions are considered for each input in each block. Therefore, the number of rules with respect the ordinary fuzzy system in estimation of \(h(X)\) and \(g(X)\) reduces from 81 to 27 and from 27 to 18, respectively. As it is known in the zero-order Takagi-Sugeno fuzzy systems, the number of tunable parameters increases with the same relation with the number of rules. The same as Example 1, initial values of the consequent part of different blocks of the HFC are randomly chosen. Initial values of states are considered as follows; \(X(0) = [0.8 0.0 0 0]^T\) for the sinusoidal signal, \(X(0) = [0.5 0.0 0 0]^T\) for the pulse signal. Simulation results are shown in Fig. 10.

Like the previous example, results illustrate the efficiency of the proposed control system. It is important to notice that like any adaptive system selection of the matrices \(P_i^{-1}\) which play the role of the learning rate is very crucial in time which is necessary for training of tunable parameters. Especially when the desired signal is pulse, proper selecting of them is very important for tracking problem due to changing behavior of the signal. Moreover, comparison of the results and the method in [8] shows the performance of proposed method especially in tracking of pulse signal.

**Example 4.** (for Theorem 2): In this example, a 4D integral-order hyper-chaotic system given by (99) is considered. In hyper-chaotic systems, the construction of the strange attractor is more complex than the chaotic attractor [30].

\[
\begin{align*}
\dot{x}_1 &= ax_1 - x_2 - u \\
\dot{x}_2 &= x_1 - x_2x_3^2 \\
\dot{x}_3 &= -b_1x_2 - b_2x_3 - b_3x_4 \\
\dot{x}_4 &= x_3 + cx_4 \\
y &= x_4 + n(t)
\end{align*}
\]  

(99)

where \(a = 0.56, b_1 = 1.0, b_2 = 1.0, b_3 = 6.0, c = 0.8\). The state space form of (99) is not in the normal form and so, does not

![Table 5](image)

**Fig. 10.** Time response of the adaptive system; proposed method \(y_{VSHFC}(\_\_)\) and method in [8] \(y_{VSHFC}(\_\_)\) follows \(y_{desire}(\_\_)\); Tracking of the sinusoidal signal (b) tracking of the pulse signal.

![Fig. 9](image)
satisfy the conditions stated in (1). With changing the coordinates of the system such that they satisfy the conditions of (1), the resulting system will be in the following form

\[ \begin{align*} 
    z_1 &= z_2 \\
    z_2 &= z_3 \\
    z_3 &= z_4 \\
    z_4 + f(Z) &= b_1 u(t) + d(t) 
\end{align*} \quad (100) \]

where

\[ f(Z) = -[(b_1 a + A)m_1 + (b_1 - b_1 B)m_2 - (b_1 + 2b_1 b_2)m_3 m_4] \]

With

\[ A = b_1 a (c - b_2), B = c^2 - b_3 - b_2 (c - b_2), C = (c^2 - b_3) - b_3 (c - b_2), \]

\[ m_1 = (c - b_2) (z_1 - z_2) + (c^2 - b_3) z_2 - z_4 \]
\[ m_2 = (c - b_2) (z_2 - z_3) + (c^2 - b_3) z_1 - z_3 \]
\[ m_3 = z_2 - z_1, m_4 = z_1. \]

Eq. (100) indicates that the resulting system is in the form of (1). The structure of the HFC control system applied in the example is depicted in Fig. 11. Numerical values of parameters of the control system are shown in Table 6. These values are achieved by trial and error.

The same as previous example, two signals are considered in to show efficiency of the proposed controller in tracking problems; sinusoidal and pulse signals. Also, three membership functions are considered for each input of each block in every simulation. Thus, every block has 9 rules and the total number of rules in HFC will be 27. On the other hand, the total number of rules and the number of tunable parameters in comparison with the ordinary fuzzy control system decrease from 81 to 27. Initial values of tunable parameters are randomly chosen. Also, the initial values of states are considered as follows; \( X(0) = [1 0 0 0]^T \) for the sinusoidal signal and \( X(0) = [0 0 0 0]^T \) for the pulse signal. Simulation results for tracking of the desire signals are shown in Fig. 12.

Tracking of the both sinusoidal signal and the pulse are done well. As it is depicted in Fig. 12, although there are sharp changes in the pulse signal the suggested controller force the system to track the desire signal with no overshoot or undershoot. Therefore, using proposed control technique tracking problem of the typical smooth and sharp signals is properly possible. Also, the results reveals performance of proposed method in comparison with the method in [8].

According to results of the examples, it can be demonstrated that the proposed control approach can guarantee all signals in the closed-loop system are bounded, and the system output tracks the desired signal even though the exact information on the nonlinear functions in controlled systems is not available.

### 6. Discussion and conclusion

In this paper, the output tracking control problem has been addressed for a class of high-order SISO nonlinear systems with a canonical structure. The nonlinear functions in systems are considered completely unknown. An additional term is added to the output control system in order to compensate uncertainties which are caused because of disturbances. Since consequent part parameters of fuzzy blocks of the hierarchical fuzzy control system except the last layer appeared in the nonlinear form in the output, the Taylor expansion is used for the linearizing them. Stability of the closed loop system is guaranteed using the Lyapunov function.

In this study, three theorems are developed. Summary of them is as follows, Theorem 1 proves how the nonlinearly parameterized tunable parameters of the HFS can be express in the linearly parameterized form using Taylor series expansion. Also, high order terms of the expansion can be demonstrate as the linearly parameterizable structure with respect to the residual term. In addition, the residual term is bounded by a linear expression with...
a known vector of functions. Due to the first theorem, Theorems 2 and 3 present the globally stable adaptive system by tuning the consequence parameters of all blocks of the HFS control system. Theorem 2 is applicable when the control gain is unity while Theorem 3 is used for the non-unity control gain systems. Generalization of the theorems to any structure of hierarchical fuzzy system and even to ordinary fuzzy systems are expressed as two corollaries.

It is important to notice that the proposed VSHF control system described in this paper differs from those described in the literature from these aspects:

(i) For the first time, the HFC system is combined with the variable structure control. Also, its nonlinearly appearing adjustable parameters are tuned using adaptation laws achieved by the Lyapunov approach.
(ii) There is an extra term in the Lyapunov approach to overcome the bounded disturbances of nonlinear systems.
(iii) In order to gain soft switching system, it is considered as a fuzzy system.

Finally, the proposed control system is applied for tracking control and regulation of three nonlinear systems; three with unity control gain, and the other with non-unity control gain. Results of simulations demonstrate the presented control algorithm is capable to control nonlinear systems with completely unknown functions in presence of bounded disturbances. Also, comparison of the results and the method in [8] shows the performance of proposed method. Moreover, in comparison with the ordinary fuzzy system, tunable parameters are significantly reduced.

Appendix A

A.1 Fuzzy switching system

The fuzzy switching system is a single input zero-order Takagi-Sugeno fuzzy system with two Gaussian membership functions, singleton fuzzifier, sum-product inference engine, and center average defuzzifier. Its input considers as sliding surface which is a metric such that determines the role of sliding surface and HFS in the control of the system. Fig. A.1 shows the structure of fuzzy switching system.

A.2 Rule base of HFS of Example 1

In the Tables A.1–A.3 the rule base of fuzzy systems in the layers of HFS in the first example are expressed. In this tables the A, B, C, D, E, F, for i = 1, ..., 3 are the Gaussian membership functions of z1, z2, z3, z4, y1, y2 respectively where y1, y2 are intermediate variables due to outputs of fuzzy systems located in the first and second layer of HFS.

<table>
<thead>
<tr>
<th>Number of rule</th>
<th>Antecedent part</th>
<th>Consequence part</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>If z₁ is A₁ &amp; z₂ is B₁</td>
<td>Then y₁ = ø₁⁰</td>
</tr>
<tr>
<td>2</td>
<td>If z₁ is A₁ &amp; z₂ is B₂</td>
<td>Then y₁ = ø₁⁰</td>
</tr>
<tr>
<td>3</td>
<td>If z₁ is A₁ &amp; z₂ is B₃</td>
<td>Then y₁ = ø₁⁰</td>
</tr>
<tr>
<td>4</td>
<td>If z₁ is A₂ &amp; z₂ is B₁</td>
<td>Then y₁ = ø₁⁰</td>
</tr>
<tr>
<td>5</td>
<td>If z₁ is A₂ &amp; z₂ is B₂</td>
<td>Then y₁ = ø₁⁰</td>
</tr>
<tr>
<td>6</td>
<td>If z₁ is A₂ &amp; z₂ is B₃</td>
<td>Then y₁ = ø₁⁰</td>
</tr>
<tr>
<td>7</td>
<td>If z₁ is A₃ &amp; z₂ is B₁</td>
<td>Then y₁ = ø₁⁰</td>
</tr>
<tr>
<td>8</td>
<td>If z₁ is A₃ &amp; z₂ is B₂</td>
<td>Then y₁ = ø₁⁰</td>
</tr>
<tr>
<td>9</td>
<td>If z₁ is A₃ &amp; z₂ is B₃</td>
<td>Then y₁ = ø₁⁰</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Number of rule</th>
<th>Antecedent part</th>
<th>Consequence part</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>If y₁ is E₁ &amp; z₁ is C₁</td>
<td>Then y₂ = ø₂⁰</td>
</tr>
<tr>
<td>2</td>
<td>If y₁ is E₁ &amp; z₁ is C₂</td>
<td>Then y₂ = ø₂⁰</td>
</tr>
<tr>
<td>3</td>
<td>If y₁ is E₁ &amp; z₁ is C₃</td>
<td>Then y₂ = ø₂⁰</td>
</tr>
<tr>
<td>4</td>
<td>If y₁ is E₂ &amp; z₁ is C₁</td>
<td>Then y₂ = ø₂⁰</td>
</tr>
<tr>
<td>5</td>
<td>If y₁ is E₂ &amp; z₁ is C₂</td>
<td>Then y₂ = ø₂⁰</td>
</tr>
<tr>
<td>6</td>
<td>If y₁ is E₂ &amp; z₁ is C₃</td>
<td>Then y₂ = ø₂⁰</td>
</tr>
<tr>
<td>7</td>
<td>If y₁ is E₃ &amp; z₁ is C₁</td>
<td>Then y₂ = ø₂⁰</td>
</tr>
<tr>
<td>8</td>
<td>If y₁ is E₃ &amp; z₁ is C₂</td>
<td>Then y₂ = ø₂⁰</td>
</tr>
<tr>
<td>9</td>
<td>If y₁ is E₃ &amp; z₁ is C₃</td>
<td>Then y₂ = ø₂⁰</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Number of rule</th>
<th>Antecedent part</th>
<th>Consequence part</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>If y₂ is F₁ &amp; z₂ is D₁</td>
<td>Then y₃ = ø₃⁰</td>
</tr>
<tr>
<td>2</td>
<td>If y₂ is F₁ &amp; z₂ is D₂</td>
<td>Then y₃ = ø₃⁰</td>
</tr>
<tr>
<td>3</td>
<td>If y₂ is F₁ &amp; z₂ is D₃</td>
<td>Then y₃ = ø₃⁰</td>
</tr>
<tr>
<td>4</td>
<td>If y₂ is F₂ &amp; z₂ is D₁</td>
<td>Then y₃ = ø₃⁰</td>
</tr>
<tr>
<td>5</td>
<td>If y₂ is F₂ &amp; z₂ is D₂</td>
<td>Then y₃ = ø₃⁰</td>
</tr>
<tr>
<td>6</td>
<td>If y₂ is F₂ &amp; z₂ is D₃</td>
<td>Then y₃ = ø₃⁰</td>
</tr>
<tr>
<td>7</td>
<td>If y₂ is F₃ &amp; z₂ is D₁</td>
<td>Then y₃ = ø₃⁰</td>
</tr>
<tr>
<td>8</td>
<td>If y₂ is F₃ &amp; z₂ is D₂</td>
<td>Then y₃ = ø₃⁰</td>
</tr>
<tr>
<td>9</td>
<td>If y₂ is F₃ &amp; z₂ is D₃</td>
<td>Then y₃ = ø₃⁰</td>
</tr>
</tbody>
</table>

Fig. A.1. Schematic of fuzzy switching system (a) Membership functions of the input (b) Block diagram of fuzzy switching system.

Please cite this article as: Mansouri M, et al. Adaptive variable structure hierarchical fuzzy control for a class of high-order nonlinear dynamic systems. ISA Transactions (2014), http://dx.doi.org/10.1016/j.isatra.2014.11.014
show the comprehensiveness of the suggested method. The result is replaced with HFS in the proposed control structure in order to first example

A.3 Simulation result of applying conventional fuzzy system to the example

In this section, using Corollary 1 the conventional fuzzy system is replaced with HFS in the proposed control structure in order to show the comprehensiveness of the suggested method. The result is shown in Fig. A.2. As it is shown in Fig. B.1 with considering all the initial values and parameters of controller and the flexible joint system as Example 1, the result is exactly the same. The only difference is in the number of rules which reduce from 81 rule in conventional fuzzy system to 27 rule in HFS.

References

[27] Zahiripour SA, Jalali AA. Designing an optimal proportional-integral sliding surface for a quarter car active suspension system with suspension components possessing uncertain constants and nonlinear characteristics. NASHRTHAYI-I MUHANDISI-I BARQ VA MUHANDISI-I KAMPYUTAR-I 2012;10:2.

Please cite this article as: Mansouri M, et al. Adaptive variable structure hierarchical fuzzy control for a class of high-order nonlinear dynamic systems. ISA Transactions (2014), http://dx.doi.org/10.1016/j.isatra.2014.11.014